Lecture 4 - Section 5.3

# ☆ A Gallery of Solution Curves of Linear Systems

For a 2D linear system $\underline{x}' = \underline{\underline{A}}\,\underline{x}$
the phase portrait depends on the eigenvectors and eigenvalues of $\underline{\underline{A}}$

There are 3 major cases:

I. Real distinct eigenvalues
II. Repeated eigenvalues
III. Complex eigenvalues.

Each case can be broken down into sub-cases
Today we will discuss all the possibilities in detail.

## I. Real distinct eigenvalues:

(a) $\lambda$ opposite signs

Ex: $\underline{x}' = \begin{bmatrix} 3 & -4 \\ 1 & 2 \end{bmatrix} \underline{x}$

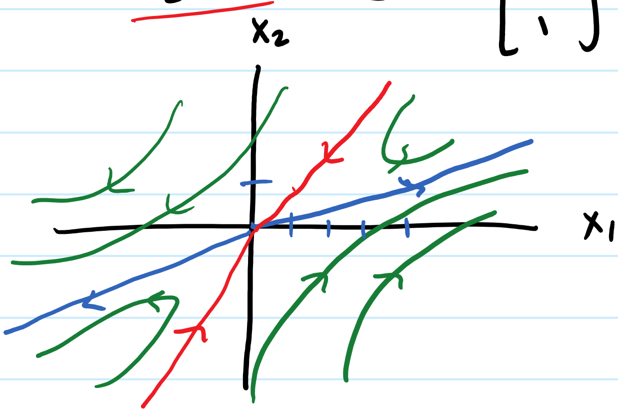
exercise: derive the eigenvalues & the eigenvectors

has $\lambda_1 = 2$ $\quad \underline{v}^{(1)} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$

$\lambda_2 = -1$ $\quad \underline{v}^{(2)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

general soln:

$$\underline{x} = c_1 e^{2t} \begin{bmatrix} 4 \\ 1 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$




$\boxed{\text{saddle point}}$

one vector points in
one vector points out

(b) $\lambda$ both negative

Ex: $\underline{x}' = \begin{bmatrix} 0 & -1 \\ 2 & -3 \end{bmatrix} \underline{x}$

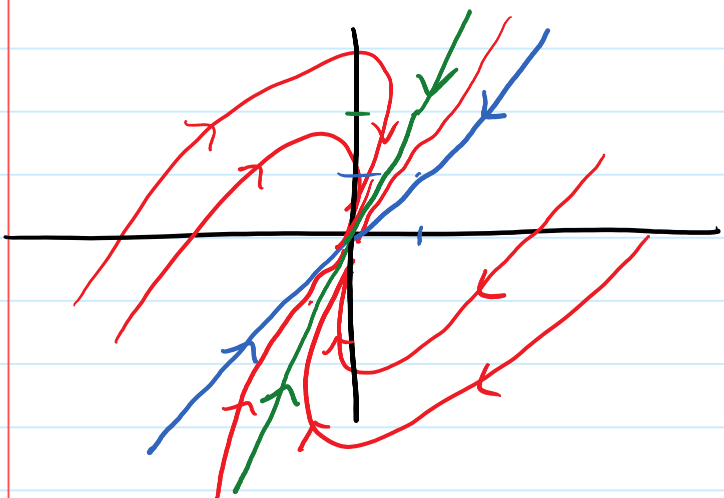
exercise: derive the eigenvalues and eigenvectors

has $\lambda_1 = -1$ $\quad \underline{v}^{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\lambda_2 = -2$ $\quad \underline{v}^{(2)} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

general soln:

$$\underline{x} = c_1 e^{-t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{-2t} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$



1. Draw eigenvectors
2. Look at limits:

as $t \to \infty$, $e^{-t}$ dominates

$\underline{x} \to$ parallel to $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

as $t \to -\infty$, $e^{-2t}$ dominates

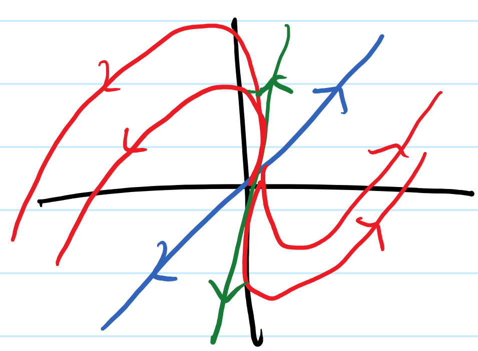
$\underline{x} \to$ parallel to $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$\boxed{\text{\underline{improper} \underline{nodal sink}}}$

multiple solutions approach the origin tangent to the same line $\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right)$

(c) $\lambda$ both positive

$\boxed{\text{\underline{improper nodal source}}}$



Same as (b) but with arrows flipped.

(d) one zero eigenvalue

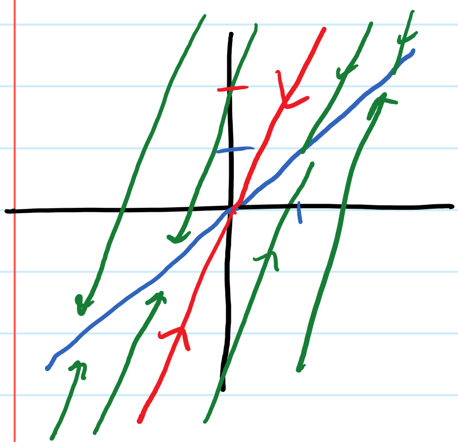
Ex: $\underline{x}' = \begin{bmatrix} 1 & -1 \\ 2 & -2 \end{bmatrix} \underline{x}$

has $\lambda_1 = 0$ $\quad \underline{v}^{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\lambda_2 = -1$ $\quad \underline{v}^{(2)} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

general solution:

$$\underline{x} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$



1. Draw eigenvectors
2. Look at limits

as $t \to \infty$ $\quad \underline{x} \to \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

as $t \to -\infty$, $e^{-t}$ dominates

$\underline{x} \to$ parallel to $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

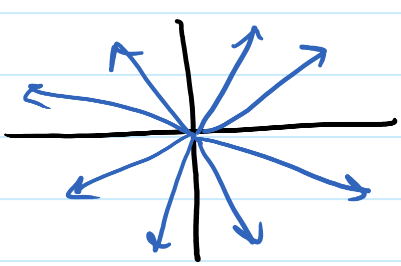
**parallel lines**

Solutions follow parallel lines, either emanating from ($\lambda_2 > 0$) or stopping on ($\lambda_2 < 0$) the eigenvector w/ $\lambda = 0$.

## II. Real repeated eigenvalues $\lambda$

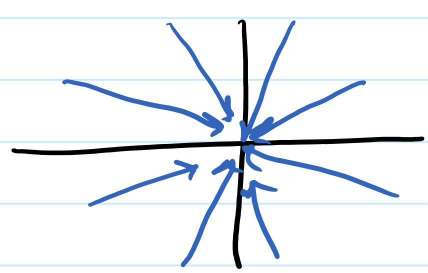
(a) $\lambda$ is complete, then

if $\lambda > 0$



proper nodal source

if $\lambda < 0$



proper nodal sink

(b) $\lambda$ is defective

$\underline{\text{Ex}}$: $\underline{x}' = \begin{bmatrix} 4 & 1 \\ -1 & 2 \end{bmatrix} \underline{x}$ $\lambda = 3$ algebraic multiplicity $= 2$

$\lambda = 3$

$\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \longrightarrow$ $v_1 + v_2 = 0$, $v_2 = -v_1$ $\underline{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

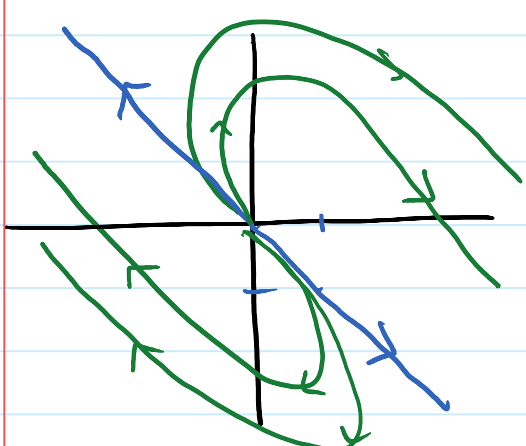
geometric multiplicity $= 1$

Find generalized eigenvector $\underline{u}$

$\begin{bmatrix} - \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ $u_1 + u_2 = 1$, $u_2 = 1 - u_1$

$u = \begin{bmatrix} u \\ 1 - u_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

General soln: $\underline{x}(t) = c_1 e^{3t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 e^{3t} \left( \begin{bmatrix} 1 \\ -1 \end{bmatrix} t + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right)$



1. Draw eigenvector $\underline{v}$
2. Look at limits

as $t \to \infty$, $te^{3t}$ dominats

$\underline{x} \to$ parallel to $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

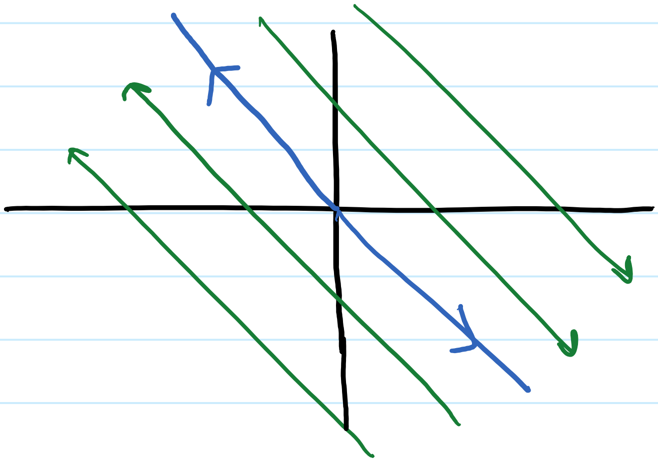
This is also an

$\boxed{\text{\underline{improper nodal source}}}$

( $\underline{\text{sink}}$ if $\lambda < 0$ )

(c) $\lambda = 0$ repeated root

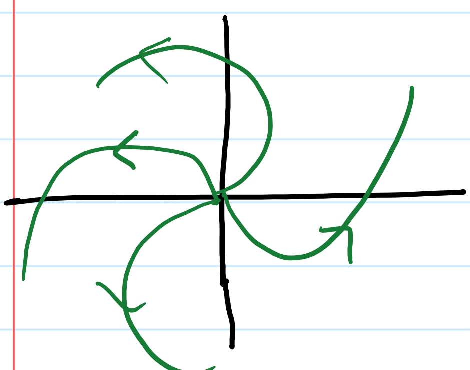
Look at example 16 in Sec 5.3 in book page 308



end up with a variation on <u>parallel lines</u>

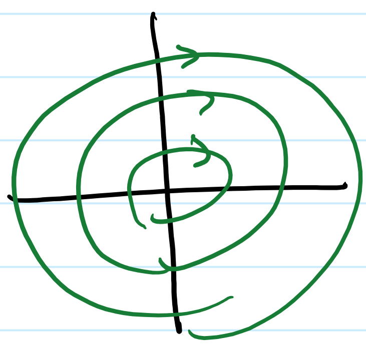
## <u>III. Complex Eigenvalues</u>:

(a) $Re(\lambda) > 0$



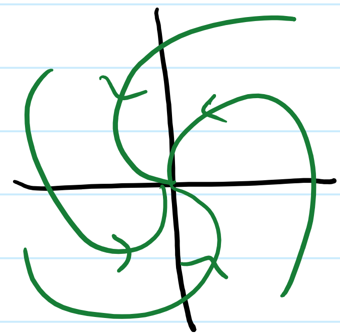
<u>spiral source</u>

(b) $Re(\lambda) = 0$



<u>center</u>

(c) $Re(\lambda) < 0$



<u>spiral sink</u>

→ look at solution to determine the direction of rotation.

i.e. $\lambda = a + bi$, $\underline{x}(t) = e^{at}(c_1 \underline{u} + c_2 \underline{v})$

then evaluate: $\begin{cases} \underline{u}(0) & \underline{v}(0) \\ \underline{u}'(0) & \underline{v}'(0) \end{cases}$